J Glob Optim (2010) 47:403–419
DOI 10.1007/s10898-008-9365-z

Fuzzy bilevel programming with multiple objectives and cooperative multiple followers

Guangquan Zhang · Jie Lu

Received: 11 September 2008 / Accepted: 3 October 2008 / Published online: 21 October 2008

Abstract Classic bilevel programming deals with two level hierarchical optimization problems in which the leader attempts to optimize his/her objective, subject to a set of constraints and his/her follower's solution. In modelling a real-world bilevel decision problem, some uncertain coefficients often appear in the objective functions and/or constraints of the leader and/or the follower. Also, the leader and the follower may have multiple conflicting objectives that should be optimized simultaneously. Furthermore, multiple followers may be involved in a decision problem and work cooperatively according to each of the possible decisions made by the leader, but with different objectives and/or constraints. Following our previous work, this study proposes a set of models to describe such fuzzy multi-objective, multi-follower (cooperative) bilevel programming problems. We then develop an approximation Kth-best algorithm to solve the problems.

Keywords Bilevel programming · Kth-best algorithm · Fuzzy sets · Optimization · Multi-objective decision making · Multi-follower bilevel programming

1 Introduction

Bilevel programming (BP) arises where decisions are made in a two level hierarchical order and each decision maker has no direct control upon the decision of the other, but actions taken by one decision maker effect returns from the other [1,2,4–6,15,16]. The decision maker at the upper level is termed the leader, and at the lower level, the follower. The leader and the follower each try to optimize their own objective function, but the decision affects the objective value at the other level [12].

G. Zhang (✉) · J. Lu
Faculty of Information Technology, University of Technology, Sydney, PO Box 123, Broadway, NSW 2007, Australia
e-mail: zhangg@it.uts.edu.au

J. Lu
e-mail: jielu@it.uts.edu.au

The majority of research on BP has centered on the linear version of the problem, i.e., linear BP problems. A set of approaches and algorithms has been well developed such as the well-known Kuhn-Tucker approach [2], Kth-best approach [3] and branch-and-bound algorithm [2,7]. In general, there are two fundamental issues in BP theory and practice. One is how to model a real-world BP problem, and the other is how to find a solution for a BP problem. Although much research has been done in the area, existing results cannot adequately model and solve a BP problem well when it corresponds to the following complex situations.

First, the decision makers at the upper level or the lower level of a bilevel decision problem may have multiple conflicting objectives which should be considered simultaneously. For example, a coordinator of a multi-division firm considers three objectives in making an aggregate production plan: maximize net profits, maximize quality of products, and maximize worker satisfaction. The three objectives could be in conflict with each other, but must be considered simultaneously. Any improvement in one objective may be achieved only at the expense of others. The one level multi-objective decision-making problem has been well researched [11], but in a bilevel decision model, the selection of an alternative solution for the leader is affected by his/her followers' optimal reactions. Therefore, how to find a solution for the leader who has multiple objectives under consideration of both their constraints and their followers' decisions needs more study.

Second, existing BP approaches mainly suppose the situation in which the objective functions and the constraints of the leader and the follower are characterized with precise coefficients/parameter. Therefore, the coefficients are required to be fixed by some values in an experimental and/or subjective manner through the experts' understanding of the nature of the coefficients in the problem-formulation process. It has been observed that, in real-world situations, the possible values of these coefficients are often only imprecisely or ambiguously known to the experts who establish the models. With this observation, it would be certainly more appropriate to interpret the experts' understanding of the coefficients as fuzzy numerical data which can be represented by means of fuzzy sets [17]. A BP problem in which the coefficients, either in objective functions or in constraints of the leader or the follower, are described by fuzzy values is called a fuzzy BP problem [13,14].

Thirdly, multiple followers may be involved in a bilevel decision. These followers may have different objectives and/or different constraints, but could work cooperatively through sharing variables in their objective and/or constraints or uncooperatively. Therefore the leader's decision will be affected not only by those followers' individual reactions but also by the relationships among these followers.

This paper deals with the above three situations together. The relationship among followers occurs in many situations which need different BP models and algorithms; this paper focuses on the situation where all followers share their decision variables, called cooperative followers [23]. In our previous study, we developed related methods to solve the fuzzy BP problem [10,18–20] and multi-follower BP problem with an uncooperative situation [8,9]. We have also conducted research on the model and solution where the leader and the follower have multiple objectives with fuzzy coefficients [21,22]. This paper aims to develop models and algorithms for fuzzy multi-objective multi-follower BP problems (FMMBP) where all followers are in a cooperative situation, called FMMBP-C problem, and focuses on a linear version of the problem.

Following the introduction, Sect. 2 gives models, related definitions, theorems, and properties for FMMBP-C problems. A general fuzzy number based approximation Kth-best algorithm for solving FMMBP-C problems is presented in Sect. 3. Two numerical examples are shown in Sect. 4 for illustrating the proposed models and algorithms. Conclusions are discussed in Sect. 5.

2 Models for fuzzy multi-objective multi-follower bilevel programming within a cooperative situation

Let R be the set of all real numbers, R^n be n-dimensional Euclidean space, and $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T \in R^n$ be any two vectors, where $x_i, y_i \in R, i = 1, 2, \ldots, n$ and Tdenotes the transpose of the vector. For any two vectors $x, y \in R^n$, we write $x \geqq y$ iff $x_i \geq y_i, \forall i = 1, 2, \ldots, n$; $x \geqq y$ iff $x \geq y$ and $x \neq y$; $x > y$ iff $x_i > y_i, \forall i = 1, 2, \ldots, n$.

Let $F(R)$ be the set of all finite fuzzy numbers. By the decomposition theorem of fuzzy sets, we have

$$\tilde{a} = \bigcup_{\lambda \in [0,1]} \lambda [a_\lambda^L, a_\lambda^R],$$

for every $\tilde{a} \in F(R)$.

Definition 2.1 Let $\tilde{a}_i \in F(R), i = 1, 2, \ldots, n$. We define $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) : \mu_{\tilde{a}} : R^n \to [0, 1]$, $x \mapsto \wedge_{i=1}^n \mu_{\tilde{a}_i}(x_i)$, where $x = (x_1, x_2, \ldots, x_n)^T \in R^n$, and $\tilde{a}$is called an n-dimensional fuzzy number on R^n.

Let $F(R^n)$ be the set of all n-dimensional finite fuzzy numbers on R^n.

Definition 2.2 For any n-dimensional fuzzy numbers $\tilde{a}, \tilde{b} \in F(R^n)$, we define

1. $\tilde{a} \succeqq \tilde{b}$ iff $a_\lambda^L \geqq b_\lambda^L$ and $a_\lambda^R \geq b_\lambda^R, \lambda \in (0, 1]$;
2. $\tilde{a} \succeq \tilde{b}$ iff $a_\lambda^L \geq b_\lambda^L$ and $a_\lambda^R \geqq b_\lambda^R, \lambda \in (0, 1]$;
3. $\tilde{a} \succ \tilde{b}$ iff $a_\lambda^L > b_\lambda^L$ and $a_\lambda^R > b_\lambda^R, \lambda \in (0, 1]$.

We call the binary relations $\succeqq$, $\succeq$ and $\succ$ a fuzzy max order, a strict fuzzy max order and a strong fuzzy max order, respectively.

A FMMBP-C problem means the followers totally share their decision variables in their objective functions and constraints. However, there are four different sub-cases within the cooperative situation which are determined by the relationships among the objective functions and constrains of the followers. We therefore develop the following four models [23]:

Model I A FMMBP-C problem in which $K(\geq 2)$ followers are involved and have shared decision variables, objective functions and constraint functions is defined as follows:

For $x \in X \subset R^n$, $y_i \in Y_i \subset R^{m_i}$, $Y = (Y_1, \ldots, Y_k)^T$, $F : X \times Y_1 \times \times \cdots \times Y_K \to F(R^s)$, $f_i : X \times Y_i \to F(R^t)$ and $i = 1, 2, \ldots, K$,

It consists of finding a solution to the upper level problem

$$\min_{x \in X} F(x, y) = \left(\tilde{c}_1^1 x + \sum_{j=1}^K \tilde{d}_{1j}^1 y_j, \tilde{c}_2^1 x + \sum_{j=1}^K \tilde{d}_{2j}^1 y_j, \ldots, \tilde{c}_s^1 x + \sum_{j=1}^K \tilde{d}_{sj}^1 y_j \right)^T$$

$$\text{subject to } \tilde{A}^1 x + \sum_{j=1}^k \tilde{B}_j^1 y_j \preceqq \tilde{b}^1$$

where $y_i (i = 1, 2, \ldots, K)$, for each value of x, is the solution of the lower level problem:

$$\min_{\substack{y_j \in Y_j \\ j=1,2,\ldots,K}} f(x, y) = \left(\tilde{c}_1^2 x + \sum_{j=1}^{K} \tilde{d}_{1j}^2 y_j, \tilde{c}_2^2 x + \sum_{j=1}^{K} \tilde{d}_{2j}^2 y_j, \ldots, \tilde{c}_t^2 x + \sum_{j=1}^{K} \tilde{d}_{tj}^2 y_j \right)^T$$

$$\text{subject to } \tilde{A}^2 x + \sum_{j=1}^{K} \tilde{B}_j^2 y_j \leqq \tilde{b}^2$$

where $\tilde{c}_i^1, \tilde{c}_j^2 \in F(R^n), \tilde{b}^1 \in F(R^p), \tilde{b}^2 \in F(R^q), \tilde{a}_{ij} \in F(R), \tilde{A}^1 = (\tilde{a}_{ij})_{p \times n}, \tilde{b}_{ij} \in F(R), \tilde{B}_z^1 = (\tilde{b}_{ij})_{p \times m}, \tilde{e}_{ij} \in F(R), \tilde{A}^2 = (\tilde{e}_{ij})_{q \times n}, \tilde{d}_{iz}^1, \tilde{d}_{jz}^2 \in F(R^m), i = 1, 2, \ldots, s, j = 1, 2, \ldots, t$, and $z = 1, 2, \ldots, K$, $\tilde{B}_z^2 = (\tilde{s}_{ij})_{q \times m}, \tilde{s}_{ij} \in F(R), z = 1, 2, \ldots, K$.

Model II A FMMBP-C problem in which $K(\geq 2)$ followers are involved and have shared decision variables and objective functions but different constraint functions is defined as follows.

For $x \in X \subset R^n, y_i \in Y_i \subset R^{m_i}, Y = (Y_1, \ldots, Y_k)^T$, $F : X \times Y_1 \times \ldots \times Y_K \to F(R^s)$, $f_i : X \times Y_i \to F(R^t)$ and $i = 1, 2, \ldots, K$.

It consists of finding a solution to the upper level problem

$$\min_{x \in X} F(x, y) = \left(\tilde{c}_1^1 x + \sum_{j=1}^{K} \tilde{d}_{1j}^1 y_j,\ \tilde{c}_2^1 x + \sum_{j=1}^{K} \tilde{d}_{2j}^1 y_j,\ \ldots,\ \tilde{c}_s^1 x + \sum_{j=1}^{K} \tilde{d}_{sj}^1 y_j \right)^T$$

$$\text{subject to } \tilde{A}^1 x + \sum_{j=1}^{k} \tilde{B}_j^1 y_j \leqq \tilde{b}^1$$

where $y_i (i = 1, 2, \ldots, K)$, for each value of x, is the solution of the lower level problem:

$$\min_{\substack{y_j \in Y_j \\ j=1,2,\ldots,K}} f(x, y) = \left(\tilde{c}_1^2 x + \sum_{j=1}^{K} \tilde{d}_{1j}^2 y_j,\ \tilde{c}_2^2 x + \sum_{j=1}^{K} \tilde{d}_{2j}^2 y_j,\ \ldots,\ \tilde{c}_t^2 x + \sum_{j=1}^{K} \tilde{d}_{tj}^2 y_j \right)^T$$

$$\text{subject to } \tilde{A}_z^2 x + \sum_{j=1}^{K} \tilde{B}_{zj}^2 y_j \leqq \tilde{b}_z^2, \quad z = 1, 2, \ldots, K,$$

where $\tilde{c}_i^{1\text{‘}}, \tilde{c}_j^2 \in F(R^n), \tilde{b}^1 \in F(R^p), \tilde{b}_z^2 \in F(R^q), \tilde{a}_{ij} \in F(R), \tilde{A}^1 = (\tilde{a}_{ij})_{p \times n}, \tilde{b}_{ij} \in F(R), \tilde{B}_z^1 = (\tilde{b}_{ij})_{p \times m}, \tilde{e}_{ij}^z \in F(R), \tilde{A}_z^2 = (\tilde{e}_{kj}^z)_{q \times n}, \tilde{d}_{iz}^1, \tilde{d}_{jz}^2 \in F(R^m), i = 1, 2, \ldots, s, j = 1, 2, \ldots, t$, and $z = 1, 2, \ldots, K$, $\tilde{B}_{zj}^2 = (\tilde{s}_{ij}^z)_{q \times m}, \tilde{s}_{ij}^z \in F(R), z = 1, 2, \ldots, K$.

Model III A FMMBP-C problem in which $K(\geq 2)$ followers are involved and have shared decision variables and constraint functions but different objective functions is defined as follows.

For $x \in X \subset R^n, y_i \in Y_i \subset R^{m_i}, Y = (Y_1, \ldots, Y_k)^T$, $F : X \times Y_1 \times \ldots \times Y_K \to F(R^s)$, $f_i : X \times Y_i \to F(R^t)$ and $i = 1, 2, \ldots, K$, It consists of finding a solution to the upper level problem

$$\min_{x\in X} F(x,y) = \left(\tilde{c}_1^1 x + \sum_{j=1}^{K}\tilde{d}_{1j}^1 y_j,\ \tilde{c}_2^1 x + \sum_{j=1}^{K}\tilde{d}_{2j}^1 y_j,\ \ldots,\ \tilde{c}_s^1 x + \sum_{j=1}^{K}\tilde{d}_{sj}^1 y_j\right)^T$$

$$\text{subject to } \tilde{A}^1 x + \sum_{j=1}^{k}\tilde{B}_j^1 y_j \precsim \tilde{b}^1$$

where $y_i (i = 1, 2, \ldots, K)$, for each value of x, is the solution of the lower level problem:

$$\min_{\substack{y_j\in Y_j\\ j=1,2,\ldots,K}} f_i(x,y) = \left(\tilde{c}_{i1}^2 x + \sum_{j=1}^{K}\tilde{d}_{i1j}^2 y_j,\ \tilde{c}_{i2}^2 x + \sum_{j=1}^{K}\tilde{d}_{i2j}^2 y_j,\ \ldots,\ \tilde{c}_{it}^2 x + \sum_{j=1}^{K}\tilde{d}_{itj}^2 y_j\right)^T$$

$$\text{subject to } \tilde{A}^2 x + \sum_{j=1}^{K}\tilde{B}_j^2 y_j \precsim \tilde{b}^2$$

where $\tilde{c}_i^{1}$, $\tilde{c}_{zj}^2 \in F(R^n)$, $\tilde{b}^1 \in F(R^p), \tilde{b}^2 \in F(R^q), \tilde{a}_{ij} \in F(R), \tilde{b}_{ij} \in F(R), \tilde{B}_z^1 = \left(\tilde{b}_{ij}\right)_{p\times m}, \tilde{e}_{ij} \in F(R), \tilde{A}^2 = \left(\tilde{e}_{ij}\right)_{q\times n}, \tilde{d}_{iz}^1,\ \tilde{d}_{gjz}^2 \in F(R^m),\ \tilde{B}_z^2 = \left(\tilde{s}_{ij}\right)_{q\times m},\ \tilde{s}_{ij} \in F(R),\ \ z = 1, 2, \ldots, K,\ \ i = 1, 2, \ldots, s,\ \ j = 1, 2, \ldots, t$, and $g, z = 1, 2, \ldots, K$.

Model IV A FMMBP-C problem in which $K(\geq 2)$ followers are involved and have shared decision variables but different objective and constraint functions is defined as follows.

For $x \in X \subset R^n, y_i \in Y_i \subset R^{m_i}, Y = (Y_1, \ldots, Y_k)^T,\ F : X \times Y_1 \times \ldots \times Y_K \rightarrow F(R^s),\ \ f_i : X \times Y_i \rightarrow F(R^t)$ and $i = 1, 2, \ldots, K$,

It consists of finding a solution to the upper level problem

$$\min_{x\in X} F(x,y) = \left(\tilde{c}_1^1 x + \sum_{j=1}^{K}\tilde{d}_{1j}^1 y_j,\ \tilde{c}_2^1 x + \sum_{j=1}^{K}\tilde{d}_{2j}^1 y_j,\ \ldots,\ \tilde{c}_s^1 x + \sum_{j=1}^{K}\tilde{d}_{sj}^1 y_j\right)^T$$

$$\text{subject to } \tilde{A}^1 x + \sum_{j=1}^{k}\tilde{B}_j^1 y_j \precsim \tilde{b}^1$$

where $y_i (i = 1, 2, \ldots, K)$, for each value of x, is the solution of the lower level problem:

$$\min_{\substack{y_J\in Y_J\\ J=1,2,\ldots,K}} f_i(x,y) = \left(\tilde{c}_{i1}^2 x + \sum_{j=1}^{K}\tilde{d}_{i1j}^2 y_j,\ \tilde{c}_{i2}^2 x + \sum_{j=1}^{K}\tilde{d}_{i2j}^2 y_j,\ \ldots,\ \tilde{c}_{it}^2 x + \sum_{j=1}^{K}\tilde{d}_{itj}^2 y_j\right)^T$$

$$\text{subject to } \tilde{A}_z^2 x + \sum_{j=1}^{K}\tilde{B}_{zj}^2 y_j \precsim \tilde{b}_z^2,\quad z = 1, 2, \ldots, K,$$

where $\tilde{c}_i^{1}$, $\tilde{c}_{zj}^2 \in F(R^n)$, $\tilde{b}^1 \in F(R^p), \tilde{b}_z^2 \in F(R^q), \tilde{a}_{ij} \in F(R), \tilde{A}^1 = \left(\tilde{a}_{ij}\right)_{p\times n},\ \tilde{b}_{ij} \in F(R), \tilde{B}_z^1 = \left(\tilde{b}_{ij}\right)_{p\times m},\ \ \tilde{e}_{ij}^z \in F(R), \tilde{A}_z^2 = \left(\tilde{e}_{kj}^z\right)_{q\times n}, \tilde{d}_{iz}^1,\ \tilde{d}_{gjz}^2 \in F(R^m),\ i = 1, 2, \ldots, s,\ \ j = 1, 2, \ldots, t$, and $g, z = 1, 2, \ldots, K, \tilde{B}_{zj}^2 = \left(\tilde{s}_{ij}^z\right)_{q\times m},\ \tilde{s}_{ij}^z \in F(R),\ z = 1, 2, \ldots, K$.

By analysing the above four models and using a weighting method, we can get a general model (**Model G**) for FMMLB-C problems as follows.

For $x \in X \subset R^n, y_i \in Y_i \subset R^{m_i}, Y = (Y_1, \ldots, Y_k)^T, \; F : X \times Y_1 \times \ldots \times Y_K \to F(R^s), \quad f_i : X \times Y_i \to F(R^t)$ and $i = 1, 2, \ldots, K$,

It consists of finding a solution to the upper level problem

$$\min_{x \in X} F(x, y) = \left(\tilde{c}_1^1 x + \sum_{j=1}^{K} \tilde{d}_{1j}^1 y_j, \; \tilde{c}_2^1 x + \sum_{j=1}^{K} \tilde{d}_{2j}^1 y_j, \; \ldots, \; \tilde{c}_s^1 x + \sum_{j=1}^{K} \tilde{d}_{sj}^1 y_j \right)^T$$

$$\text{subject to } \tilde{A}^1 x + \sum_{j=1}^{k} \tilde{B}_j^1 y_j \precsim \tilde{b}^1$$

where $y_i (i = 1, 2, \ldots, K)$, for each value of x, is the solution of the lower level problem:

$$\min_{\substack{y_j \in Y_j \\ j=1,2,\ldots,K}} f(x, y) = \left(\tilde{c}_1^2 x + \sum_{j=1}^{K} \tilde{d}_{1j}^2 y_j, \; \tilde{c}_2^2 x + \sum_{j=1}^{K} \tilde{d}_{2j}^2 y_j, \; \ldots, \; \tilde{c}_t^2 x + \sum_{j=1}^{K} \tilde{d}_{tj}^2 y_j \right)^T$$

$$\text{subject to } \tilde{A}_z^2 x + \sum_{j=1}^{K} \tilde{B}_{zj}^2 y_j \precsim \tilde{b}_z^2, \; z = 1, 2, \ldots, K,$$

where $\tilde{c}_i^{1\text{`}}, \; \tilde{c}_j^2 \in F(R^n), \; \tilde{b}^1 \in F(R^p), \tilde{b}^2 \in F(R^q), \tilde{a}_{ij} \in F(R), \tilde{A}^1 = \left(\tilde{a}_{ij}\right)_{p \times n}, \; \tilde{b}_{ij} \in F(R), \tilde{B}_z^1 = \left(\tilde{b}_{ij}\right)_{p \times m}, \; \tilde{e}_{ij} \in F(R), \tilde{d}_{iz}^1, \; \tilde{d}_{jz}^2 \in F(R^m), \; \tilde{B}_z^2 = \left(\tilde{s}_{ij}\right)_{q \times m}, \; \tilde{s}_{ij} \in F(R), \; i = 1, 2, \ldots, s, \; j = 1, 2, \ldots, t$, and $z = 1, 2, \ldots, K$.

3 An approximation K th-best algorithm

This section will present an approximation K-best algorithm for solving a FMMBP-C problem. Associated with the FMMBP-C problem, we now consider the following multi-objective linear BP (MOLBP) problem:

For $x \in X \subset R^n, y_i \in Y_i \subset R^{m_i}, Y = (Y_1, \ldots, Y_k)^T, F : X \times Y_1 \times \cdots \times Y_K \to F(R^s), f_i : X \times Y_i \to F(R^t)$ and $i = 1, 2, \ldots, K$,

$$\min_{x \in X} (F(x, y))_\lambda^{L(R)} = \left((F_1(x, y))_\lambda^L, (F_1(x, y))_\lambda^R, \ldots (F_s(x, y))_\lambda^L, \; (F_s(x, y))_\lambda^R \right)^T, \quad \lambda \in [0, 1] \tag{1a}$$

$$\text{subject to } A^1{}_\lambda^L x + \sum_{j=1}^{K} B_{j\lambda}^{1L} y_j \leqq b^1{}_\lambda^L \tag{1b}$$

$$A^1{}_\lambda^R x + \sum_{j=1}^{K} B_{j\lambda}^{1R} y_j \leqq b^1{}_\lambda^R, \lambda \in [0, 1]$$

$$\min_{y \in Y} (f(x, y))_\lambda^{L(R)} = \left((f_1(x, y))_\lambda^L, (f_1(x, y))_\lambda^R, \ldots, \; (f_t(x, y))_\lambda^L, (f_t(x, y))_\lambda^R \right)^T, \quad \lambda \in [0, 1] \tag{1c}$$

$$\text{subject to } A^{2L}_{z\lambda}x + \sum_{j=1}^{K} B^{2\ L}_{zj\lambda}y_j \leqq b^{2L}_{z\lambda}, \tag{1d}$$

$$A^{2L}_{z\lambda}x + \sum_{j=1}^{K} B^{2\ L}_{zj\lambda}y_j \ \leqq b^{2L}_{z\lambda}, \quad z = 1, 2, \ldots, K, \lambda \in [0, 1]$$

where $(F_i(x, y))^L_\lambda = c^{1L}_{i\lambda}x + \sum_{j=1}^{K} d^{1^L}_{ij\lambda}y_j$, $(F_i(x, y))^R_\lambda = c^{1^R}_{i\lambda}x + \sum_{j=1}^{K} d^{1^R}_{ij\lambda}y_j$, $\left(f^z_j(x, y)\right)^L_\lambda = c^{2^L}_{jz\lambda}x + \sum_{j=1}^{K} d^{2^L}_{zij\lambda}y_j$ *and* $\left(f^z_j(x, y)\right)^R_\lambda = c^{2^R}_{jz\lambda}x + \sum_{j=1}^{K} d^{2^R}_{zij\lambda}y_j$, $\lambda \in [0, 1], c^{1^R}_{i_\lambda}, c^{1^R}_{i_\lambda}, c^{2^L}_{jz_\lambda}, c^{2^R}_{jz_\lambda} \in R^n, d_{ij\lambda}{}^{1L}, d_{ij\lambda}1^R,\ d^{2L}_{izj\lambda},\ d_{izj\lambda}2^R{}_\lambda{}^R \in R^m,\ \ b_\lambda 1^L,\ b_\lambda{}^{1R} \in R^p,\ \ b_{z\lambda}2^L,\ b^{2^R}_{z\lambda} \in R^q,\ \ A^1{}^L_\lambda = a_{ij}{}^L_\lambda,\ A^1{}^R_\lambda = {}_ij^R_\lambda \in\in R^{p\times n},\ \ A^{2^L}_{z\lambda} = e_{izj}{}^L_\lambda,\ A_{z\lambda}2^R = e_{izj}{}^R_\lambda \in\ R^{q\times n}$, $B_{z\lambda}1^L = b_{izj}{}^L_\lambda,\ B_{z\lambda}1^R = b_{izj}{}^R_\lambda \in R^{p\times m},\ B^{2L}_{zj\lambda} = \left(S^{zj}_{ir}\right)^L_\lambda$
$B^{2^R}_{zj\lambda} = \left(S^{zj}_{ir}\right)^R_\lambda \in R^{q\times m},\ i = 1, 2,\ \ldots,\ s,\ \ j = 1, 2,\ \ldots,\ t,\ z = 1, 2, \ldots, K.$

By using Definition 2.2, we have

Theorem 3.1 *Let $(x^*,\ y^*)$ be the optimal solution of the MOLBP problem defined by (1) . Then it is also an optimal solution of the FMMBP-C problem defined by Model G.*

We can use the theory of solving fuzzy MOLBP [19] to find an optimal solution for Model G because Model G is a fuzzy MOLBP problem.

Theorem 3.2 *[22] For $x \in X \subset R^n, y_i \in Y_i \subset R^{m_i}, I = 1, 2, \ldots, K$, if all the fuzzy coefficients have piecewise trapezoidal membership functions in Model G,*

$$\mu_{\tilde z}(t) = \begin{cases} 0 & t < z^L_{\alpha_0} \\ \frac{\alpha_1-\alpha_0}{z^L_{\alpha_1}-z^L_{\alpha_0}}\left(t - z^L_{\alpha_0}\right) + \alpha_0 & z^L_{\alpha_0} \leqq t < z^L_{\alpha_1} \\ \frac{\alpha_1-\alpha_0}{z^L_{\alpha_2}-z^L_{\alpha_1}}\left(t - z^L_{\alpha_1}\right) + \alpha_1 & z^L_{\alpha_1} \leqq t < z^L_{\alpha_2} \\ \quad\cdots & \quad\cdots \\ \alpha & z^L_{\alpha_n} \leqq t < z^R_{\alpha_n} \\ \frac{\alpha_n-\alpha_{n-1}}{z^R_{\alpha_{n-1}}-z^R_{\alpha_n}}\left(-t + z^R_{\alpha_{n-1}}\right) + \alpha_{n-1} & z^R_{\alpha_n} \leq t < z^R_{\alpha_{n-1}} \\ \quad\cdots & \quad\cdots \\ \dfrac{\alpha_0 - \alpha_1}{z^R_{\alpha_1} - z^R_{\alpha_0}}\left(-t + z^R_{\alpha_0}\right) + \alpha_0 & z^R_{\alpha_1} \leqq t \leqq z^R_{\alpha_0} \\ 0 & z^R_{\alpha_0} < t \end{cases} \tag{2}$$

where $\tilde z$ denotes any fuzzy coefficients in Model G, then, $(x^*,\ y^*)$ is a complete optimal solution to the FMMBP-C problem if and only if $(x^*,\ y^*)$ is an optimal solution to the MOLBP problem:

$$\min_{x\in X}(F_i(x, y))^L_{\alpha_j} = c^{1^L}_{i\alpha_j}x + \sum_{z=1}^{K} d^{1^L}_{iz\alpha_j}y_z, \quad i = 1, \ldots, s, \quad j = 0, 1, \ldots, n \tag{3a}$$

$$\min_{x\in X}(F_i(x, y))^R_{\alpha_j} = c^{1^R}_{i\alpha_j}x + \sum_{z=1}^{K} d^{1^R}_{iz\alpha_j}y_z, \quad i = 1, \ldots, s,\ \ j = 0, 1, \ldots, n$$

$$\textit{subject to } A^{1^L}_{\alpha_j} x + \sum_{z=1}^{K} B^{1^L}_{z\alpha_j} y_z \leqq b^{1^L}_{\alpha_j}, \quad j = 0, 1, \ldots, n \tag{3b}$$

$$A^{1^R}_{\alpha_j} x + \sum_{z=1}^{K} B^{1^R}_{z\alpha_j} y_z \leqq b^{1^R}_{\alpha_j}, \quad j = 0, 1, \ldots, n$$

$$\min_{y \in Y} \left(f_i^z(x, y)\right)^L_{\alpha_j} = c^{2^L}_{iz\alpha_j} x + \sum_{k=1}^{K} d^{2^L}_{izk\alpha_j} y_k, \quad i = 1, \ldots, t, \; j = 0, 1, \ldots, n$$

$$\min_{y \in Y} \left(f_i^z(x, y)\right)^R_{\alpha_j} = c^{2^R}_{iz\alpha_j} x + \sum_{k=1}^{K} d^{2^R}_{izk\alpha_j} y_k, \quad z = 1, \ldots, K, \tag{3c}$$

$$\textit{subject to } A^{2^L}_{z\alpha_j} x + \sum_{z=1}^{K} B^{2^L}_{zz\alpha_j} y_z \leqq b^{2^L}_{z\alpha_j}, \quad j = 0, 1, \ldots, n \tag{3d}$$

$$A^{2^R}_{z_{\alpha_j}} x + \sum_{z=1}^{K} B^{2^R}_{zz\alpha_j} y_z \leqq b^{2^R}_{z\alpha_j}, \quad z = 1, 2, \ldots, K$$

We note

$$\bar{A}_1 x + \bar{B}_1 y \leqq \bar{b}_1 \tag{3'b}$$

$$\bar{A}_2 x + \bar{B}_2 y \leqq \bar{b}_2 \tag{3'd}$$

where

$$\bar{A}_1 = \begin{pmatrix} A^{1^L}_{\alpha_0} \\ \vdots \\ A^{1^L}_{\alpha_n} \\ A^{1^R}_{\alpha_0} \\ \vdots \\ A^{1^R}_{\alpha_n} \end{pmatrix}, \quad \bar{A}_2 = \begin{pmatrix} A^{2^L}_{1\alpha_0} \\ \vdots \\ A^{2^L}_{K\alpha_n} \\ A^{2^R}_{1\alpha_0} \\ \vdots \\ A^{2^R}_{K\alpha_n} \end{pmatrix}, \quad \bar{B}_1 = \begin{pmatrix} \sum_{z=1}^{K} B^{1^L}_{z\alpha_0} \\ \vdots \\ \sum_{z=1}^{K} B^{1^L}_{z\alpha_n} \\ \sum_{z=1}^{K} B^{1^R}_{z\alpha_0} \\ \vdots \\ \sum_{z=1}^{K} B^{1^L}_{z\alpha_n} \end{pmatrix}, \quad \bar{B}_2 = \begin{pmatrix} \sum_{z=1}^{K} B^{2^L}_{z1\alpha_0} \\ \vdots \\ \sum_{z=1}^{K} B^{2^L}_{zK\alpha_n} \\ \sum_{z=1}^{K} B^{2^R}_{z1\alpha_0} \\ \vdots \\ \sum_{z=1}^{K} B^{2^R}_{zK\alpha_n} \end{pmatrix},$$

$$\bar{b}_1 = \begin{pmatrix} b^{1^L}_{\alpha_0} \\ \vdots \\ b^{1^L}_{\alpha_n} \\ b^{1^R}_{\alpha_0} \\ \vdots \\ b^{1^R}_{\alpha_n} \end{pmatrix}, \quad \bar{b}_2 = \begin{pmatrix} b^{2^L}_{1\alpha_0} \\ \vdots \\ b^{2^L}_{K\alpha_n} \\ b^{2^R}_{1\alpha_0} \\ \vdots \\ b^{2^L}_{K\alpha_n} \end{pmatrix}.$$

Then we can re-write (3a–3d) by using

$$\min_{x\in X}\ (F_i(x,y))^L_{\alpha_j} = c^{1^L}_{i\alpha_j}x + \sum_{z=1}^{K} d^{1^L}_{iz\alpha_j}y_z,\quad i=1,\ldots,s,\ j=0,\quad 1,\ldots,n \tag{3'a}$$

$$\min_{x\in X}\ (F_i(x,y))^R_{\alpha_j} = c^{1^R}_{i\alpha_j}x + \sum_{z=1}^{K} d^{1^R}_{iz\alpha_j}y_z,\quad i=1,\ldots,s,\ j=0,\quad 1,\ldots,n$$

$$\textit{subject to } \bar{A}_1x + \bar{B}_1y \leqq \bar{b}_1, \tag{3'b}$$

$$\min_{y\in Y}\ \left(f_i^z(x,y)\right)^L_{\alpha_j} = c^{2^L}_{iz\alpha_j}x + \sum_{k=1}^{K} d^{2^L}_{izk\alpha_j}y_k, i=1,\ldots,t,\ j=0,\quad 1,\ldots,n$$

$$\min_{y\in Y}\ \left(f_i^z(x,y)\right)^R_{\alpha_j} = c^{2^R}_{iz\alpha_j}x + \sum_{k=1}^{K} d^{2^R}_{izk\alpha_j}y_k,\quad z=1,\ldots,K, \tag{3'c}$$

$$\text{subject to } \bar{A}_2x + \bar{B}_2y \leqq \bar{b}_2. \tag{3'd}$$

To solve the FMMBP-C problem, we need to solve its transformed form (3′). For solving (3′), we can use the method of weighting [13] to this problem, such that it becomes the following problem:

$$\min_{x\in X}(F(x,y)) = \sum_{j=1}^{s} w_{j1}\left(\sum_{i=0}^{n}\left(c^{1^L}_{j_{\alpha i}}x + \sum_{k=1}^{K} d^{1^L}_{jk_{\alpha i}}y_k\right)\right.$$
$$\left. + \sum_{i=0}^{n}\left(c^{1^R}_{j_{\alpha i}}x + \sum_{k=1}^{K} d^{1^R}_{jk_{\alpha i}}y_k\right)\right) \tag{4a}$$

$$\text{subject to } \bar{A}_1x + \bar{B}_1y \leqq \bar{b}_1, \tag{4b}$$

$$\min_{y\in Y}(f(x,y)) = \sum_{z=1}^{K} w_z \sum_{j=1}^{t} w_{j2}\left(\sum_{i=0}^{n}\left(c^{2^L}_{jz_{\alpha i}}x + \sum_{k=1}^{K} d^{2^L}_{jzk_{\alpha i}}y_k\right)\right.$$
$$\left. + \sum_{i=0}^{n}\left(c^{2^R}_{jz_{\alpha i}}x + \sum_{k=1}^{K} d^{2^R}_{jzk_{\alpha i}}y_k\right)\right) \tag{4c}$$

$$\text{subject to } \bar{A}_2x + \bar{B}_2y \leqq \bar{b}_2 \tag{4d}$$

In order to get a solution for above problem (4), we give a definition of optimal solution and related theorems as follows.

Definition 3.1 (a) Constraint region of the linear BP problem:

$$S = \left\{(x,y) : x\in X, y\in Y, \bar{A}_1x + \bar{B}_1y \leqq \bar{b}_1, \bar{A}_2x + \bar{B}_2y \leqq \bar{b}_2\right\}$$

(b) Feasible set for the follower for each fixed$x \in X$:

$$S(x) = \left\{ y \in Y : \bar{B}_2 y \leqq \bar{b}_2 - \bar{A}_2 x \right\}$$

(c) Projection of S onto the leader's decision space:

$$S(X) = \left\{ x \in X : \exists y \in Y, \bar{A}_1 x + \bar{B}_1 y \leqq \bar{b}_1, \bar{A}_2 x + \bar{B}_2 y \leqq \bar{b}_2 \right\}$$

(d) Follower's rational reaction set for$x \in S(X)$:

$$P(x) = \left\{ y \in Y : y \in \arg\min[\left(f(x, \hat{y})\right) : \hat{y} \in S(x)] \right\}$$

where $\arg\min[f(x, \hat{y}) : \hat{y} \in S(x)] = \{y \in S(x) : (f(x, y)) \leqq \left(f(x, \hat{y})\right), \hat{y} \in S(x)\}$
Inducible region:

$$IR = \{(x, y) : (x, y) \in S, y \in P(x)\}$$

The rational reaction set $P(x)$ defines the response while the inducible region IR represents the set over which the leader may optimize his/her objective. Thus in terms of the above notations, the linear BP problem can be written as

$$\min\{F(x, y) : (x, y) \in IR\}. \tag{5}$$

Theorem 3.3 *The inducible region can be written equivalently as piecewise linear equality constraint comprised of supporting hyper planes of constraint region S.*

Proof Let us begin by writing the inducible region of Definition 3.1(e) explicitly as follows:

$$IR = \left\{ (x, y) : (x, y) \in S, \bar{d}_2 y = \min[\bar{d}_2 \tilde{y} : \bar{B}_2 \tilde{y} \leqq \bar{b}_2 - \bar{A}_2 x, \ \tilde{y} \geq 0] \right\} \tag{6}$$

Now we define

$$Q(x) = \min \left\{ \bar{d}_2 y : \bar{B}_2 y \leqq \bar{b}_2 - \bar{A}_2 x, y \geq 0 \right\}. \tag{7}$$

For each value of $x \in S(X)$, the resulting feasible region to problem (4) is nonempty and compact. Thus $Q(x)$, which is a linear program parameterized in x, always has a solution. From duality theory, we get

$$\max \left\{ u(\bar{A}_2 x - b_2) : u\bar{B}_2 \geqq -\bar{d}_2, u \geq 0 \right\}, \tag{8}$$

which has the same optimal value as(7) at the solution u^*. Let u^1,..., u^s be a listing of all the vertices of the constraint region of(8) given by $U = u : u\bar{B}_2 \geqq -\bar{d}_2, u \geq 0$ Because we know that a solution to(8) occurs at a vertex of U, we get the equivalent problem

$$\max \left\{ u^j(\bar{A}_2 x - \bar{b}_2) : u^j \in \left\{ u^1, \ldots, u^s \right\} \right\}, \tag{9}$$

which demonstrates that $Q(x)$ is a piecewise linear function. Rewriting IR as

$$IR = \left\{ (x, y) \in S : Q(x) - \bar{d}_2 y = 0 \right\}, \tag{10}$$

yields the desired result. □

By this theorem, we give the following corollaries:

Corollary 3.1 *The linear BP problem (4) is equivalent to minimizing F over a feasible region comprised of a piecewise linear equality constraint.*

Proof From (5) and Theorem 3.2, we have the desired result. □

Corollary 3.2 *A solution for the linear BP problem occurs at a vertex of IR.*

Proof A linear BP programming can be written(5). Since $F$is linear, if a solution exists, one must occur at a vertex ofIR. The proof is completed □

Now, we give a very important theorem which is the core for proposing an approximation Kth-best approach.

Theorem 3.4 *The solution $(x^*, y^*)$of the linear BP problem occurs at a vertex of S.*

Proof Let $(x^1, y^1), \ldots, (x^r, y^r)$ be the distinct vertices of S. Since any point in S can be written a convex combination of these vertices, let $(x^*, y^*) = \sum_{i=1}^{r} \alpha_i (x^i, y^i)$, where $\sum_{i=1}^{r} \alpha_i = 1, \alpha_i \geq 0, i = 1, \ldots, \bar{r}$ and $\bar{r} \leq r$. It must be shown that$\bar{r} = 1$. To see this let us write the constraints to(4) at (x^*, y^*) in their piecewise linear form(10).

$$\begin{aligned} 0 &= Q(x^*) - \bar{d}_2 y^* \\ &= Q\left(\sum_i \alpha_i x^i\right) - \bar{d}_2 \left(\sum_i \alpha_i y^i\right) \\ &\leq \sum_i \alpha_i Q(x^i) - \sum_i \alpha_i \bar{d}_2 y^i \end{aligned}$$

by convexity of $Q(x)$,

$$= \sum_i \alpha_i (Q(x^i) - \bar{d}_2 y^i).$$

But by definition,

$$Q(x^i) = \min_{y \in S(x^i)} \bar{d}_2 y \leq \bar{d}_2 y^i.$$

Therefore, $Q(x^i) - \bar{d}_2 y^i \leq 0, i = 1, \ldots, \bar{r}$. Noting that $\alpha_i \geq 0, i = 1, \ldots, \bar{r}$, the equality in the preceding expression must hold or else a contradiction would result in the sequence above. Consequently, $Q(x^i) - \bar{d}_2 y^i = 0$ for all i. This implies that $(x^i, y^i) \in IR, i = 1, \ldots, \bar{r}$ and (x^*, y^*)can be written as a convex combination of points in IR. Because (x^*, y^*) is a vertex of IR, a contradiction results unless $\bar{r} = 1$. □

We therefore give the following corollary.

Corollary 3.3 *If x is an extreme point of IR, it is an extreme point of S.*

Proof Let (x^*, y^*) be an extreme point of IR and assume that it is not an extreme point of S. Let
$(x^1, y^1), \ldots, (x^r, y^r)$ be the distinct vertices of S. Since any point in S can be written a convex combination of these vertices, let $(x^*, y^*) = \sum_{i=1}^{r} \alpha_i (x^i, y^i)$, where $\sum_{i=1}^{r} \alpha_i = 1, \alpha_i \geq 0, i = 1, \ldots, \bar{r}$ and $\bar{r} \leq r$. It must be shown that $\bar{r} = 1$. To see this let us write the constraints to (4) at (x^*, y^*) in their piecewise linear form (10).

$$
\begin{aligned}
0 &= Q(x^*) - \bar{d}_2 y^* \\
&= Q\left(\sum_i \alpha_i x^i\right) - \bar{d}_2\left(\sum_i \alpha_i y^i\right) \\
&\leq \sum_i \alpha_i Q(x^i) - \sum_i \alpha_i \bar{d}_2 y^i
\end{aligned}
$$

by convexity of $Q(x)$,

$$= \sum_i \alpha_i (Q(x^i) - \bar{d}_2 y^i).$$

But by definition,

$$Q(x^i) = \min{}_{y} \in S(x^i)\bar{d}_2 y \leq \bar{d}_2 y^i.$$

Therefore, $Q(x^i) - \bar{d}_2 y^i \leq 0, i = 1, \ldots, \bar{r}$. Noting that $\alpha_i \geq 0, i = 1, \ldots, \bar{r}$, the equality in the preceding expression must hold or else a contradiction would result in the sequence above. Consequently, $Q(x^i) - \bar{d}_2 y^i = 0$ for all i. This implies that $(x^i, y^i) \in IR, i = 1, \ldots, \bar{r}$ and (x^*, y^*)can be written as a convex combination of points in IR. Because (x^*, y^*) is an extreme point of IR, a contradiction results unless $\bar{r} = 1$. This means that (x^*, y^*) is an extreme point of S. The proof is completed. □

Theorem 3.2 and Corollary 3.3 have provided the theoretical foundation for our new approach to solve FMMBP-C problems. It means that by searching extreme points on the constraint region S, we can efficiently find an optimal solution for a linear BP problem. The basic idea of our extended properties approach is that according to the objective function of the upper level, we descendent order all the extreme points on S, and select the first extreme point to check if it is on the inducible region IR. If yes, the current extreme point is the optimal solution. Otherwise, we select the next one and check it.

More specifically, let $(x_{[1]}, y_{[1]}), \ldots, (x_{[N]}, y_{[N]})$ denote the N ordered extreme points to the linear programming problem

$$\min\{\bar{c}_1 x + \bar{d}_1 y : (x, y) \in S\}, \tag{11}$$

such that

$$\bar{c}_1 x_{[i]} + \bar{d}_1 y_{[i]} \leq \bar{c}_1 x_{[i+1]} + \bar{d}_1 y_{[i+1]}, \quad i = 1, \ldots, N-1,$$

where

$$\bar{c}_1 = \sum_{j=1}^{s} w_{j1} \sum_{i=0}^{n} \left(c_{j\alpha i}^{1^L} + c_{j\alpha i}^{1^R}\right),\ \bar{d}_1 = \sum_{j=1}^{s} w_{j1} \sum_{i=0}^{n} \sum_{k=1}^{K} \left(d_{jk\alpha i}^{1^L} + d_{jk\alpha i}^{1^R}\right).$$

Let $\tilde{y}$ denote the optimal solution to the following problem

$$\min(f(x_{[i]}, y) : y \in S(x_{[i]})). \tag{12}$$

We only need to find the smallest i $(i \in \{1, \ldots, N\})$ under which $y_{[i]} = \tilde{y}$.

Let us write (12) as follows

$$
\begin{aligned}
&\min f(x, y) \\
&\text{subject to } y \in S(x) \\
&x = x_{[i]}.
\end{aligned}
$$

From Definition 3.1(a) and (c), we have

$$\min f(x, y) = \bar{c}_2 x + \bar{d}_2 y \tag{13a}$$
$$\text{subject to } \bar{A}_1 x + \bar{B}_1 y \le \bar{b}_1 \tag{13b}$$
$$\bar{A}_2 x + \bar{B}_2 y \le \bar{b}_2 \tag{13c}$$
$$x = x_{[i]} \tag{13d}$$
$$y \ge 0, \tag{13e}$$

where

$$\bar{c}_2 = \sum_{z=1}^{K} w_z \sum_{j=1}^{t} w_{j1} \sum_{i=0}^{n} (c_{jz\alpha i}^{2L} + c_{jz\alpha i}^{2R}),$$
$$\bar{d}_2 = \sum_{z=1}^{K} w_z \sum_{j=1}^{t} w_{j1} \sum_{i=0}^{n} \sum_{k=1}^{K} (d_{jkz\alpha i}^{2L} + d_{jkz\alpha i}^{2R}).$$

To solve(13), we need to select one ordered extreme point$(x_{[i]}, y_{[i]})$, and then solve(13) to obtain the optimal solution$\tilde{y}$. If $\tilde{y} = y_{[i]}$, $(x_{[i]}, y_{[i]})$ is the global optimum to(4). Otherwise, check next extreme point.

Based on above definition of optimal solution and Theorem 3.4, we propose an approximation Kth-best approach for solving FMMBP-C problem as follows.

The approximation K th-best approach:

Step 1 Given weights $w_{j1}(j = 1, 2, \ldots, s)$ and $w_{j2}(j = 1, 2, \ldots, t)$ for the objectives of the leader and the follower and weights $(k = 1, 2, \ldots, K)$ for all followers, respectively and let

$$\sum_{j=1}^{s} w_{j1} = 1, \quad \sum_{j=1}^{t} w_{j2} = 1, \quad \sum_{k=1}^{K} w_k = 1.$$

Step 2 Transform the problem FMMBP-C to the problem (3')

Step 3 Set $l = 1$ and a range of errors $\epsilon > 0$.

Step 4 Let the interval [0, 1] be decomposed into 2^{l-1} equal sub-intervals with $(2^{l-1}+1)$ nodes
$\lambda_i \left(i = 0, \ldots, 2^{l-1}\right)$ which are arranged in the order of $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{2^{l-1}} = 1$

Step 5 Transform the problem(3) to the problem(4) by the weighting method and solve $(\text{MOLBP})_{2^l}$ i.e.(4) by using the Kth-best approach [10] for obtaining an optimal solution
$(x, \ y)_{2^l}$.

Step 6 Put $i \leftarrow 1$. Solve (11) with the simplex method to obtain the optimal solution $(x_{[1]}, y_{[1]})$. Let $W = \{(x_{[1]}, y_{[1]})\}$ and $T = \phi$. Go to Step 7.

Step 7 Solve (13) with the bounded simplex method. Let $\tilde{y}$ denote the optimal solution to (13). If $\tilde{y} = y_{[i]}$, stop; $(x_{[i]}, y_{[i]})$ is the global optimum to (4) with $K^* = i$. Otherwise, go to Step 8.

Step 8 Let $W_{[i]}$ denote the set of adjacent extreme points of $(x_{[i]}, y_{[i]})$ such that $(x, y) \in W_{[i]}$ implies $\bar{c}_1 x + \bar{d}_1 y \ge \bar{c}_1 x_{[i]} + \bar{d}_1 y_{[i]}$. Let $T = T \cup \{(x_{[i]}, y_{[i]})\}$ and $W = (W \cup W_{[i]}) \setminus T$. Go to Step 9.

Step 9 Set $i \leftarrow i+1$ and choose $(x_{[i]}, y_{[i]})$ so that $f x_{[i]} + g y_{[i]} = \min\{\bar{c}_1 x + \bar{d}_1 y : (x, y) \in W\}$. Go to Step 7.

Step 10 $l = l+1$, repeat Step 4 to Step 9 to solve $(\text{MOLBP})_2^{l+1}$.

Step 11 If $\left\| (x,\ y)_{2^{l+1}} - (x,\ y)_{2^l} \right\| < \epsilon$, then the solution $(x^*,\ y^*)$ of the FMMBP-C problem is $(x,\ y)_{2^{l+1}}$, otherwise, update l to $2l$ and go back to Step 10.

Step 12 Show the result of the FMMBP-C problem, stop.

4 Numerical examples

We first apply the proposed approximation Kth-best algorithm to solve a simple FMMBP-C problem, and then for a complex problem.

Example 1 Let $x = (x_1,\ x_2)^T \in R^2$ be the leaders' decision variables, and with two objectives F^1 and F^2, $y_1 = (y_1,\ y_2,\ y_3)^T \in R^3$ and $y_2 = (y_1,\ y_2,\ y_3)^T \in R^3$ be two followers' decision variables with their objective f_1 and f_2 respectively, and $X = \{x \geq 0\}, Y = \{y \geq 0\}$. We can build the following model for the decision problem:

$$
\begin{aligned}
&\max_{x \in X} F^1(x, y) = \left(-\tilde{1},\ \tilde{3}\right)(x_1,\ x_2)^T + \left(\tilde{2},\ \tilde{6},\ -\tilde{1}\right)(y_1,\ y_2,\ y_3)^T \\
&\max_{x \in X} F^2(x, y) = \left(\tilde{2},\ \tilde{1}\right)(x_1,\ x_2)^T + \left(\tilde{3},\ \tilde{2}\right)(y_1,\ y_2)^T \\
&\text{subject to } \left(\tilde{2},\ -\tilde{1}\right)(x_1,\ x_2)^T + \left(\tilde{3},\ \tilde{2},\ -\tilde{1}\right)(y_1,\ y_2,\ y_3)^T \leq \tilde{2} \\
&\left(\tilde{1},\ \tilde{3}\right)(x_1,\ x_2)^T + \left(-\tilde{1},\ \tilde{0}\right)(y_1,\ y_2)^T \leq \tilde{1} \\
&\max_{y \in Y} f_1(x, y) = \left(\tilde{3},\ \tilde{3}\right)(x_1,\ x_2)^T + \left(\tilde{2},\ \tilde{1},\ \tilde{2}\right)(y_1,\ y_2,\ y_3)^T \\
&\text{subject to } \left(-\tilde{8},\ \tilde{8}\right)(x_1,\ x_2)^T + \left(\tilde{2},\ \tilde{0},\ -\tilde{3}\right)(y_1,\ y_2,\ y_3)^T \leq \tilde{0} \\
&\left(\tilde{2},\ -\tilde{8}\right)(x_1,\ x_2)^T + \left(\tilde{2},\ \tilde{4},\ \tilde{0}\right)(y_1,\ y_2,\ y_3)^T \leq \tilde{4} \\
&\max_{y \in Y} f_2(x, y) = \left(\tilde{1},\ \tilde{3}\right)(x_1,\ x_2)^T + \left(\tilde{3},\ \tilde{2},\ \tilde{3}\right)(y_1,\ y_2,\ y_3)^T \\
&\text{subject to } \left(-\tilde{8},\ \tilde{8}\right)(x_1,\ x_2)^T + \left(\tilde{2},\ \tilde{0},\ -\tilde{3}\right)(y_1,\ y_2,\ y_3)^T \leq \tilde{0} \\
&\left(\tilde{2},\ -\tilde{8}\right)(x_1,\ x_2)^T + \left(\tilde{2},\ \tilde{4},\ \tilde{0}\right)(y_1,\ y_2,\ y_3)^T \leq \tilde{4}
\end{aligned}
$$

where

$$
\mu_{\tilde{1}}(t) = \begin{cases} 0 & t < 0 \\ t^2 & 0 \leqq t < 1 \\ 2 - t & 1 \leqq t < 2 \\ 0 & 2 \leqq t \end{cases},\quad
\mu_{\tilde{2}}(t) = \begin{cases} 0 & t < 1 \\ t - 1 & 1 \leqq t < 2 \\ 3 - t & 2 \leqq t < 3 \\ 0 & 3 \leqq t \end{cases},\quad
\mu_{\tilde{3}}(t) = \begin{cases} 0 & t < 2 \\ t - 2 & 2 \leqq t < 3 \\ 4 - t & 3 \leqq t < 4 \\ 0 & 4 \leqq t \end{cases},
$$

$$
\mu_{\tilde{4}}(t) = \begin{cases} 0 & t < 3 \\ t - 3 & 3 \leqq t < 4 \\ 5 - t & 4 \leqq t < 5 \\ 0 & 5 \leqq t \end{cases},\quad
\mu_{\tilde{0}}(t) = \begin{cases} 0 & t < -1 \\ t+1 & -1 \leqq t < 0 \\ 1-t^2 & 0 \leqq t < 1 \\ 0 & 1 \leqq t \end{cases}.\quad
\mu_{\tilde{6}}(t) = \begin{cases} 0 & t < 5 \\ t - 5 & 5 \leqq t < 6 \\ 7 - t & 6 \leqq t < 7 \\ 0 & 7 \leqq t \end{cases},
$$

$$
\mu_{\tilde{8}}(t) = \begin{cases} 0 & t < 7 \\ t - 7 & 7 \leqq t < 8 \\ 9 - t & 8 \leqq t < 9 \\ 0 & 9 \leqq t \end{cases},
$$

Obvious, this is a FMMBP-C problem in which two followers sharing decision variables have the same constraints but different objective functions. It is the problem described in Model III. It can be transformed into Model G by using a weighting method in the followers' objective functions. It then can be solved by using the proposed approximation Kth-best algorithm which has been implemented in a bilevel decision support system.

The solution of the problem is $\left(x_1^*,\ x_2^*,\ y_1^*,\ y_2^*,\ y_3^*\right)$ =(0, 1.3696, 0.5652, 0.2464, 0) with

$$\max_{x\in X} F^1(x,y) = \left(-\tilde{1},\ \tilde{3}\right)(0,\ 1.3696)^T + \left(\tilde{2},\ \tilde{6},\ -\tilde{1}\right)(0.5652,\ 0.2464,\ 0)^T$$

$$\max_{x\in X} F^2(x,y) = \left(\tilde{2},\ \tilde{1}\right)(0,\ 1.3696)^T + \left(\tilde{3},\ \tilde{2}\right)(0.5652,\ 0.2464)^T$$

$$\max_{y\in Y} f_1(x,y) = \left(\tilde{3},\ \tilde{3}\right)(0,\ 1.3696)^T + \left(\tilde{2},\ \tilde{1},\ \tilde{2}\right)(0.5652,\ 0.2464,\ 0)^T$$

$$\max_{y\in Y} f_2(x,y) = \left(\tilde{1},\ \tilde{3}\right)(0,\ 1.3696)^T + \left(\tilde{3},\ \tilde{2},\ \tilde{3}\right)(0.5652,\ 0.2464,\ 0)^T$$

Example 2 Let $x = (x_1,\ x_2)^T \in R^2$ be the leader's decision variables, F^1 and F^2 be the leader's objective; $y_1 = (y_1,\ y_2,\ y_3)^T \in R^3$ and $y_2 = (y_1,\ y_2,\ y_3)^T \in R^3$ be two followers' decision variables and f_1^1, f_1^2 and f_2^1, f_2^2 be the followers' objectives respectively, $X = \{x \geq 0\}$, $Y = \{y \geq 0\}$. We can build the following model for the bilevel decision problem:

$$\max_{x\in X} F^1(x,y) = \left(\tilde{6},\ \tilde{3}\right)(x_1,\ x_2)^T + \left(\tilde{8},\ \tilde{6},\ \tilde{1}\right)(y_1,\ y_2,\ y_3)^T$$

$$\max_{x\in X} F^2(x,y) = \left(\tilde{4},\ -\tilde{3}\right)(x_1,\ x_2)^T + \left(-\tilde{1},\ \tilde{2},\ \tilde{6}\right)(y_1,\ y_2)^T$$

$$\text{subject to } \left(\tilde{1},\ \tilde{3}\right)(x_1,\ x_2)^T + \left(-\tilde{1},\ \tilde{0}\right)(y_1,\ y_3)^T \leq \tilde{1}$$

$$\max_{y\in Y} f_1^1(x,y) = \left(\tilde{3},\ \tilde{3}\right)(x_1,\ x_2)^T + \left(\tilde{2},\ \tilde{0},\ \tilde{2}\right)(y_1,\ y_2,\ y_3)^T$$
$$\max_{y\in Y} f_1^2(x,y) = \left(\tilde{2},\ -\tilde{1}\right)(x_1,\ x_2)^T + \left(\tilde{4},\ \tilde{3},\ -\tilde{1}\right)(y_1,\ y_2,\ y_3)^T$$

$$\text{subject to } \left(-\tilde{8},\ \tilde{8}\right)(x_1,\ x_2)^T + \left(\tilde{2},\ \tilde{0},\ -\tilde{3}\right)(y_1,\ y_2,\ y_3)^T \leq \tilde{0}$$

$$\max_{y\in Y} f_2^1(x,y) = \left(\tilde{2},\ \tilde{1}\right)(x_1,\ x_2)^T + \left(\tilde{3},\ \tilde{1},\ \tilde{2}\right)(y_1,\ y_2,\ y_3)^T$$

$$\max_{y\in Y} f_2^2(x,y) = \left(-\tilde{1},\ \tilde{3}\right)(x_1,\ x_2)^T + \left(\tilde{2},\ -\tilde{1},\ \tilde{2}\right)(y_1,\ y_2,\ y_3)^T$$

$$\text{subject to } -\tilde{8}x_2 + \left(\tilde{2},\ \tilde{4},\ \tilde{0}\right)(y_1,\ y_2,\ y_3)^T \leq \tilde{4}$$

where

$$\mu_{\tilde{1}}(t)=\begin{cases}0 & t<0\\ t^2 & 0\leqq t<1\\ 2-t & 1\leqq t<2\\ 0 & 2\leqq t\end{cases},\ \mu_{\tilde{2}}(t)=\begin{cases}0 & t<1\\ t-1 & 1\leqq t<2\\ 3-t & 2\leqq t<3\\ 0 & 3\leqq t\end{cases},\ \mu_{\tilde{3}}(t)=\begin{cases}0 & t<2\\ t-2 & 2\leqq t<3\\ 4-t & 3\leqq t<4\\ 0 & 4\leqq t\end{cases},$$

$$\mu_{\tilde{4}}(t)=\begin{cases}0 & t<3\\ t-3 & 3\leqq t<4\\ 5-t & 4\leqq t<5\\ 0 & 5\leqq t\end{cases},\ \mu_{\tilde{0}}(t)=\begin{cases}0 & t<-1\\ t+1 & -1\leqq t<0\\ 1-t^2 & 0\leqq t<1\\ 0 & 1\leqq t\end{cases}.\ \mu_{\tilde{6}}(t)=\begin{cases}0 & t<5\\ t-5 & 5\leqq t<6\\ 7-t & 6\leqq t<7\\ 0 & 7\leqq t\end{cases},$$

$$\mu_{\tilde{8}}(t)=\begin{cases}0 & t<7\\ t-7 & 7\leqq t<8\\ 9-t & 8\leqq t<9\\ 0 & 9\leqq t\end{cases},$$

Obviously, this is a FMMBP-C problem under Model IV as there are shared decision variables but different objectives and constraints among the two followers. Now, we first transform it to Model G by using a weighting method. We then use the proposed approximation Kth-best algorithm to get a solution.

The solution of the problem is $\left(x_1^*,\ x_2^*,\ y_1^*,\ y_2^*,\ y_3^*\right)$ =(0, 0.5273, 0, 0.6, 2.8909) with

$$\begin{aligned}
\max_{x\in X} F^1(x,y) &= \left(\tilde{6},\ \tilde{3}\right)(0,\ 0.5273)^T+\left(\tilde{8},\ \tilde{6},\ \tilde{1}\right)(0,\ 0.6,\ 2.8909)^T\\
\max_{x\in X} F^2(x,y) &= \left(\tilde{4},\ -\tilde{3}\right)(0,\ 0.5273)^T+\left(-\tilde{1},\ \tilde{2},\ \tilde{6}\right)(0,\ 0.6,\ 2.8909)^T\\
\max_{y\in Y} f_1^1(x,y) &= \left(\tilde{3},\ \tilde{3}\right)(0,\ 0.5273)^T+\left(\tilde{2},\ \tilde{0},\ \tilde{2}\right)(0,\ 0.6,\ 2.8909)^T\\
\max_{y\in Y} f_1^2(x,y) &= \left(\tilde{2},\ -\tilde{1}\right)(0,\ 0.5273)^T+\left(\tilde{4},\ \tilde{3},\ -\tilde{1}\right)(0,\ 0.6,\ 2.8909)^T\\
\max_{y\in Y} f_2^1(x,y) &= \left(\tilde{2},\ \tilde{1}\right)(0,\ 0.5273)^T+\left(\tilde{3},\ \tilde{1},\ \tilde{2}\right)(0,\ 0.6,\ 2.8909)^T\\
\max_{y\in Y} f_2^2(x,y) &= \left(-\tilde{1},\ \tilde{3}\right)(0,\ 0.5273)^T+\left(\tilde{2},\ -\tilde{1},\ \tilde{2}\right)(0,\ 0.6,\ 2.8909)^T
\end{aligned}$$

These two examples illustrate how to model a FMMBP-C problem by using proposed models and obtain a solution by using the approximation Kth-best algorithm.

5 Conclusions

A real-world bilevel decision problem may be modelled to have multiple objective functions, fuzzy coefficients, and multiple followers. The research deals with the three issues together when followers are in a cooperative situation. This paper proposes a fuzzy number based approximation Kth-best algorithm to solve this complex problem. Further study includes the development of models and algorithms for other situations among followers, such as when followers do not share or only partially share their decision variables. We will also explore effective applications of the proposed techniques.

Acknowledgment The work presented in this paper was supported by the Australian Research Council (ARC) under discovery grant DP0557154.

References

1. Anandalingam, G., Friesz, T.: Hierarchical optimization: an introduction. Ann. Oper. Res. **34**, 1–11 (1992). doi:10.1007/BF02098169
2. Bard, J.: Practical Bilevel Optimization: Algorithms and Applications. Kluwer, Amsterdam (1998)
3. Bialas, W., Karwan, M.: Twolevel linear programming. Manage. Sci. **30**, 1004–1020 (1984)
4. Bracken, J., McGill, J.: Mathematical programs with optimization problems in the constraints. Oper. Res. **21**, 37–44 (1973)
5. Candler, W., Townsley, R.: A linear twolevel programming problem. Comput. Oper. Res. **9**, 59–76 (1982). doi:10.1016/0305-0548(82)90006-5
6. Dempe, S.: A simple algorithm for the linear bilevel programming problem. Optimization **18**, 373–385 (1987). doi:10.1080/02331938708843247
7. Hansen, P., Jaumard, B., Savard, G.: New branchandbound rules for linear bilevel programming. SIAM J. Sci. Stat. Comput. **13**, 1194–1217 (1992). doi:10.1137/0913069
8. Lu, J., Shi, C., Zhang, G.: On bilevel multi-follower decision-making: general framework and solutions. Inf. Sci. **176**, 1607–1627 (2006). doi:10.1016/j.ins.2005.04.010
9. Lu, J., Shi, C., Zhang, G.: An extended branch-and-bound algorithm for bilevel multi-follower decision making in a referential-uncooperative situation. Int. J. Inf. Technol. Decis. Mak. **6**, 1–18 (2007)
10. Lu, J., Zhang, G., Dillon, T.: Fuzzy multi-objective bilevel decision making by an approximation Kth-best approach. J. Multi-Valued Logic Soft Comput. **14**(3–5), 205–232 (2008)
11. Lu, J., Zhang, G., Ruan, D., Wu, F.: Multi-Objective Group Decision Making: Methods, Software and Applications with Fuzzy Set Technology. Imperial College Press, London (2007)
12. Miller, T., Friesz, T., Tobin, R.: Heuristic algorithms for delivered price spatially competitive network facility location problems. Ann. Oper. Res. **34**, 177–202 (1992). doi:10.1007/BF02098179
13. Sakawa, M.: Fussy Sets and Interactive Multi-Objective Optimization. Plenum, New York (1993)
14. Sakawa, M., Nishizaki, I., Uemura, Y.: Interactive fuzzy programming for multilevel linear programming problems with fuzzy parameters. Fuzzy Sets Syst. **109**, 3–19 (2000). doi:10.1016/S0165-0114(98)00130-4
15. Von Stackelberg, H.: The theory of the market economy. Oxford University Press, New York (1952)
16. White, D., Anandalingam, G.: A penalty function approach for solving bi-level linear programs. J. Glob. Optim. **3**, 397–419 (1993). doi:10.1007/BF01096412
17. Zadeh L.A.: Fuzzy Sets. Inf. Contr. **8**, 338–353 (1965). doi:10.1016/S0019-9958(65)90241-X
18. Zhang, G., Lu, J.: The definition of optimal solution and an extended Kuhn-Tucker approach for fuzzy linear bilevel programming. IEEE Comput. Intell. Bull. **5**, 1–7 (2005)
19. Zhang, G., Lu, J.: Model and approach of fuzzy bilevel decision making for logistics planning problem. J. Enterprise Inf. Manage. **20**, 178–197 (2007). doi:10.1108/17410390710725760
20. Zhang, G., Lu, J., Dillon, T. : A branch-and-bound algorithm for fuzzy bilevel decision making. In: D. Ruan (ed.) Applied Artificial Intelligence, pp. 291–298. World Scientific, Singapore (2006)
21. Zhang, G., Lu, J., Dillon, T.: Decentralized multi-objective bilevel decision making with fuzzy demands. Knowl. Base. Syst. **20**, 495–507 (2007). doi:10.1016/j.knosys.2007.01.003
22. Zhang, G., Lu, J., Dillon, T. et al. : Solution concepts and an approximation Kuhn-Tucker approach for fuzzy multi-objective linear bilevel programming. In: P.M. Pardalos (ed.) Pareto Optimality, Game Theory and Equilibria, pp. 457–480. Spinger, New York (2008)
23. Zhang, G., Lu, J., Gao, Y.: Fuzzy bilevel programming: Multi-objective and multi-follower with shared variables. Int. J. Uncertainty Fuzz. Knowledge-Based Syst. **16**(2), 105–133 (2008)